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Phase-covariant quantum cloning

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Abstract

Quantum cloning machines for equatorial qubits are studied. For a 1 to 2 phase-covariant quantum cloning machine, using Hilbert–Schmidt norm and Bures fidelity, we show that our transformation can achieve the bound of the fidelity.

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1. Introduction

Quantum computing and quantum information have been attracting a great deal of interest. They differ in many aspects from the classical theories. One of the most fundamental differences between classical and quantum information is the no-cloning theorem [1]. It tells us that arbitrary quantum information cannot be copied exactly. The no-cloning theorem for pure states is also extended to the case where a general mixed state cannot be broadcast [2]. Recently, a stronger no-cloning theorem [3] has been proposed which combines the no-cloning theorem [1] and no-deleting theorem [4]. However, the no-cloning theorem does not forbid imperfect cloning. And several kinds of quantum cloning machines (QCMs) have been proposed, such as the universal QCM [5], probabilistic QCM [6] and asymmetric QCM [7]. The optimal fidelity and transformations of the universal QCMs are found in [5, 8–14]. Recently, an experiment on universal QCM was reported in [15].

In the proof of the no-cloning theorem, Wootters and Zurek introduced a QCM which has the property that the quality of the copy it makes depends on the input states [1]. To diminish or cancel this disadvantage, Bužek and Hillery proposed a universal quantum cloning machine (UQCM) for arbitrary pure states where the copying process is input state independent. They use the Hilbert–Schmidt norm to quantify distances between the input density operator and the output density operators. Bruß *et al* [8] discussed the performance of a UQCM by analysing the role of the symmetry and isotropy conditions imposed on the system and found the optimal UQCM and the optimal state-dependent quantum cloning. Optimal fidelity and optimal quantum cloning transformations of the general N to M $(M > N)$ case are presented in [10–13]. The relation between quantum cloning and superluminal signalling is proposed and discussed in [16, 17]. It was also shown that the UQCM can be realized by a network consisting of quantum gates [18].

In the case of UQCM, the input states are arbitrary pure states. In this paper, we study the QCM for a restricted set of pure input states. The Bloch vector is restricted to the intersection of the x–z plane with the Bloch sphere. These kinds of qubits are the so-called equatorial qubits [19] and the corresponding QCM is called phase-covariant quantum cloning. Applying the method of Bužek and Hillery [5], we propose a possible extension of the original transformation. We demand that (I) the density matrices of the two output states be the same, and that (II) the distance between the input density operator and the output density operators be input state independent. To evaluate the distance between two states, we use both the Hilbert–Schmidt norm and Bures fidelity. There is a family of transformations which satisfies the above two conditions. At a special point, we can obtain an optimal fidelity. The correspondent transformation for the x–z equator agrees with the results of Bruß *et al* [19] who studied the optimal quantum cloning for equatorial qubits by taking BB84 [20] states as input. The fidelity of quantum cloning for the equatorial qubits is higher than the original Bužek and Hillery UQCM [5] because we already know part of the information of the input states. This is expected as the more information about the input is given, the better one can clone each of its states.

In most cases of quantum computation and quantum information study, we generally suppose that we can produce enough identical copies of a quantum state. However, it is different in quantum cryptography. If a cheater (Eve) intends to eavesdrop on the quantum states, a QCM will provide her with the most direct method. BB84 quantum key distribution has been attracting a great deal of interest. Because only four states instead of six states are used in this quantum key distribution, the optimal method eavesdropping is to use the phase-covariant QCM instead of the well-known UQCM [19]. And further, Eve needs to know the quality of her copy by various criteria. In this paper, we use two distinguishability measures to quantify the quality of the cloning machine. For each distinguishability measure, we compare both reduced density operators and the whole density operators between input and output. And we show that these two distinguishability measures give the same result.

The paper is organized as follows. In section 2, we introduce the transformation for the equator in the x–z plane. In section 3, we use the Hilbert–Schmidt norm to evaluate the distance between the input state and output states, and the minimal distance is found. In section 4, we use the Bures metric to define the fidelity, and the condition of orientation invariance of the Bloch vector is studied.

2. Transformation

Instead of arbitrary input states, we consider the input state which we intend to clone as a restricted set of states. It is a pure superposition state,

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1}$$

with $\alpha^2 + \beta^2 = 1$. Here, we use an assumption that α and β are real in contrast to complex when we consider the case of UQCM. This means that the y component of the Bloch vector of the input qubits is zero. Because there is just one unknown parameter in the input state under consideration, we expect that we can achieve a better quality in quantum cloning if we can find an appropriate phase-covariant QCM.

In order to have a better quality in phase-covariant quantum cloning than the UQCM, we need a different cloning transformation. We propose the following transformation,

$$\begin{aligned} |0\rangle_{a_1}|Q\rangle_{a_2a_3} &\to (|0\rangle_{a_1}|0\rangle_{a_2} + \lambda|1\rangle_{a_1}|1\rangle_{a_2})|Q_0\rangle_{a_3} + (|1\rangle_{a_1}|0\rangle_{a_2} + |0\rangle_{a_1}|1\rangle_{a_2})|Y_0\rangle_{a_3} \\ |1\rangle_{a_1}|Q\rangle_{a_2a_3} &\to (|1\rangle_{a_1}|1\rangle_{a_2} + \lambda|0\rangle_{a_1}|0\rangle_{a_2})|Q_1\rangle_{a_3} + (|1\rangle_{a_1}|0\rangle_{a_2} + |0\rangle_{a_1}|1\rangle_{a_2})|Y_1\rangle_{a_3} \end{aligned} \tag{2}$$

where the states $|Q_j\rangle_{a_3}, |Y_j\rangle_{a_3}$, $j = 0, 1$ are not necessarily orthonormal. We will sometimes drop the subscript a_3 for convenience. Explicitly, this transformation is a generalization of the original one proposed by Bužek and Hillery [5]. When $\lambda = 0$, this transformation is reduced to the original transformation. Here, we remark that it is still unclear whether the cloning transformation presented above can achieve the optimal point if we choose appropriate parameters, because the supposed cloning transformation (2) is not the most general one. We shall show in the following sections that this cloning transformation indeed can achieve the optimal point. For convenience, we restrict λ to be real and $\lambda \neq \pm 1$. We also assume

$$\langle Q_0|Q_1\rangle = \langle Q_1|Q_0\rangle = 0. \tag{3}$$

Considering the unitarity of the transformation, we have the following relations:

$$(1+\lambda^2)\langle Q_j|Q_j\rangle + 2\langle Y_j|Y_j\rangle = 1 \qquad j = 0, 1 \tag{4}$$

$$\langle Y_0|Y_1\rangle = \langle Y_1|Y_0\rangle = 0. \tag{5}$$

As proposed by Bužek and Hillery, we further assume the following relations to reduce the free parameters:

$$\langle Q_j|Y_j\rangle = 0 \qquad j = 0, 1 \tag{6}$$

$$\langle Y_0|Y_0\rangle = \langle Y_1|Y_1\rangle \equiv \xi \tag{7}$$

$$\langle Y_0|Q_1\rangle = \langle Q_0|Y_1\rangle = \langle Q_1|Y_0\rangle = \langle Y_1|Q_0\rangle \equiv \frac{\eta}{2}. \tag{8}$$

For simplicity, we shall use the following standard notation

$$|jk\rangle = |j\rangle_{a_1}|k\rangle_{a_2} \qquad j, k = 0, 1 \tag{9}$$

and

$$|+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle). \tag{10}$$

Obviously, $|\pm\rangle$ and $|00\rangle, |11\rangle$ constitute an orthonormal basis.

The output density operator $\rho_{ab}^{(\mathrm{out})}$ describing the output state after the copying procedure reads

$$\begin{aligned}\rho_{a_1a_2}^{(\mathrm{out})} = {}& |00\rangle\langle 00| \left\{\frac{1-2\xi}{1+\lambda^2}[\lambda^2 + \alpha^2(1-\lambda^2)]\right\} + (|00\rangle\langle 10| + |00\rangle\langle 01| + |11\rangle\langle 10| \\ & + |11\rangle\langle 01| + |01\rangle\langle 00| + |10\rangle\langle 00| + |01\rangle\langle 11| + |10\rangle\langle 11|)\left[\frac{\eta}{2}\alpha\beta(\lambda+1)\right] \\ & + (|00\rangle\langle 11| + |11\rangle\langle 00|)\left(\frac{1-2\xi}{1+\lambda^2}\lambda\right) + \xi(|01\rangle\langle 10| + |01\rangle\langle 01| + |10\rangle\langle 10| \\ & + |10\rangle\langle 01|) + |11\rangle\langle 11|\left\{\frac{1-2\xi}{1+\lambda^2}[\alpha^2(\lambda^2-1)+1]\right\}\end{aligned} \tag{11}$$

where $\rho_{a_1a_2}^{(\mathrm{out})} = \mathrm{Tr}_{a_3}\left[\rho_{a_1a_2a_3}^{(\mathrm{out})}\right]$ with $\rho_{a_1a_2a_3}^{(\mathrm{out})} \equiv |\Psi\rangle_{a_1a_2a_3}^{(\mathrm{out})}\,{}_{a_1a_2a_3}^{(\mathrm{out})}\langle\Psi|$. Taking the trace over mode a_2 or mode a_1, we get the reduced density operator for mode a_1 or mode a_2, $\rho_{a_1}^{(\mathrm{out})}$ or $\rho_{a_2}^{(\mathrm{out})}$,

$$\begin{aligned}\rho_{a_1}^{(\mathrm{out})} = \rho_{a_2}^{(\mathrm{out})} = {}& |0\rangle\langle 0|\left((\alpha^2+\lambda^2\beta^2)\frac{1-2\xi}{1+\lambda^2} + \xi\right)(|0\rangle\langle 1| + |1\rangle\langle 0|)\alpha\beta\eta(1+\lambda) \\ & + |1\rangle\langle 1|\left(\xi + (\beta^2+\lambda^2\alpha^2)\frac{1-2\xi}{1+\lambda^2}\right).\end{aligned} \tag{12}$$

We see that the output density operators $\rho_{a_1}^{(\mathrm{out})}$ and $\rho_{a_2}^{(\mathrm{out})}$ are exactly the same. However, it is well known that they are not equal to the original input density operator. Next, we first use the Hilbert–Schmidt norm to evaluate the distance between the input density operator and output density operators.

3. Hilbert–Schmidt norm

For two-dimensional space, the Hilbert–Schmidt norm is believed to give a reasonable result in comparing density matrices though it becomes less appropriate for finite-dimensional spaces as the dimension increases. The Hilbert–Schmidt norm defines the distance between the input density operator and output density operator as

$$D_a \equiv \mathrm{Tr}\left[\rho_a^{(\mathrm{out})} - \rho_a^{(\mathrm{in})}\right]^2 \tag{13}$$

where $\rho_a^{(\mathrm{in})}$ is the input density operator. The distance between the two-mode density operators $\rho_{a_1a_2}^{(\mathrm{out})}$ and $\rho_{a_1a_2}^{(\mathrm{in})} = \rho_{a_1}^{(\mathrm{in})} \otimes \rho_{a_1}^{(\mathrm{in})}$ which corresponds to the ideal copy is defined as

$$D_{a_1a_2}^{(2)} = \mathrm{Tr}\left[\rho_{a_1a_2}^{(\mathrm{out})} - \rho_{a_1a_2}^{(\mathrm{in})}\right]^2. \tag{14}$$

With the help of relation (12), we find

$$D_a = \left\{\xi + \frac{1-2\xi}{1+\lambda^2}[\alpha^2(1-\lambda^2)+\lambda^2] - \alpha^2\right\}^2 + 2\alpha^2(1-\alpha^2)(\lambda\eta+\eta-1)^2 + \left\{\xi - 1 + \frac{1-2\xi}{1+\lambda^2}[1+\alpha^2(\lambda^2-1)] + \alpha^2\right\}^2. \tag{15}$$

We demand that this distance be independent of the parameter α^2. This means that the quality of the copies it makes is independent of the input state:

$$\frac{\partial}{\partial\alpha^2} D_a = 0. \tag{16}$$

We can choose the following solution:

$$\eta = \frac{1-\lambda}{1+\lambda^2}(1-2\xi). \tag{17}$$

Thus, we get

$$D_a = 2\left(\xi\frac{1-\lambda^2}{1+\lambda^2} + \frac{\lambda^2}{1+\lambda^2}\right)^2. \tag{18}$$

In the case $\lambda = 0$, we find $\eta = 1 - 2\xi$ and $D_a = 2\xi^2$. These are exactly the original results obtained by Bužek and Hillery [5].

In order to calculate $D_{a_1a_2}^{(2)}$, we can rewrite the output density operator $\rho_{a_1a_2}^{(\mathrm{out})}$ by choosing the basis in (10). Substituting relation (17) into the two-mode output density operator, we obtain

$$\rho_{a_1a_2}^{(\mathrm{out})} = |00\rangle\langle 00|\left\{\frac{1-2\xi}{1+\lambda^2}[\lambda^2+\alpha^2(1-\lambda^2)]\right\} + (|00\rangle\langle +| + |+\rangle\langle 00| + |11\rangle\langle +| + |+\rangle\langle 11|)\left\{\sqrt{2}\alpha\beta\frac{1-\lambda^2}{2(1+\lambda^2)}(1-2\xi)\right\} + (|00\rangle\langle 11| + |11\rangle\langle 00|)\left\{\frac{1-2\xi}{1+\lambda^2}\lambda\right\} + 2\xi|+\rangle\langle +| + |11\rangle\langle 11|\left\{\frac{1-2\xi}{1+\lambda^2}[\alpha^2(\lambda^2-1)+1]\right\}. \tag{19}$$

By straightforward calculations, we also have

$$\rho^{(\mathrm{in})}_{a_1a_2} = \alpha^4|00\rangle\langle 00| + \sqrt{2}\alpha^3\beta(|00\rangle\langle +| + |+\rangle\langle 00|) + \alpha^2\beta^2(|00\rangle\langle 11| + |11\rangle\langle 00|) + 2\alpha^2\beta^2|+\rangle\langle +| + \sqrt{2}\alpha\beta^3(|+\rangle\langle 11| + |11\rangle\langle +|) + \beta^4|11\rangle\langle 11|. \tag{20}$$

And with definition (14), we obtain

$$D^{(2)}_{a_1a_2} = (U_{11})^2 + (U_{22})^2 + (U_{33})^2 + 2(U_{12})^2 + 2(U_{13})^2 + 2(U_{23})^2 \tag{21}$$

where

$$U_{11} = \alpha^4 - \frac{1-2\xi}{1+\lambda^2}[\lambda^2 + \alpha^2(1-\lambda^2)] \qquad U_{22} = 2\xi - 2\alpha^2 + 2\alpha^4$$
$$U_{33} = \alpha^4 - 2\alpha^2 + 1 - \frac{1-2\xi}{1+\lambda^2}[\alpha^2(\lambda^2-1)+1] \qquad U_{12} = \sqrt{2}\alpha\beta\left[\alpha^2 - \frac{1-\lambda^2}{1+\lambda^2}\left(\frac{1}{2}-\xi\right)\right]$$
$$U_{13} = \alpha^2\beta^2 - \frac{1-2\xi}{1+\lambda^2}\lambda \qquad U_{23} = \sqrt{2}\alpha\beta\left[\beta^2 - \frac{1-\lambda^2}{1+\lambda^2}\left(\frac{1}{2}-\xi\right)\right]. \tag{22}$$

We still impose the condition

$$\frac{\partial}{\partial\alpha^2}D^{(2)}_{a_1a_2} = 0 \tag{23}$$

and then

$$\xi = \frac{(1-\lambda)^2}{2(3-2\lambda+3\lambda^2)}. \tag{24}$$

Substitution of these results into D_a and $D^{(2)}_{a_1a_2}$ gives

$$D_a = \frac{(1-2\lambda+5\lambda^2)^2}{2(3-2\lambda+3\lambda^2)^2} \qquad D^{(2)}_{a_1a_2} = \frac{2(1-4\lambda+12\lambda^2-8\lambda^3+7\lambda^4)}{(3-2\lambda+3\lambda^2)^2}. \tag{25}$$

Therefore, we can have a family of transformations which satisfy the two conditions (I) and (II). In the case $\lambda = 0$, we recover the Bužek and Hillery result,

$$D_a = \tfrac{1}{18} \approx 0.056 \qquad D^{(2)}_{a_1a_2} = \tfrac{2}{9} \approx 0.22. \tag{26}$$

Our aim is to find smaller D_a and $D^{(2)}_{a_1a_2}$ for equatorial qubits, and to prove that the corresponding cloning transformation is the optimal QCM. We can show that in the region $0 < \lambda < 1/3$, both D_a and $D^{(2)}_{ab}$ take smaller values than in the case $\lambda = 0$. When we choose

$$\lambda = 3 - 2\sqrt{2} \tag{27}$$

both D_a and $D^{(2)}_{a_1a_2}$ take their minimal values,

$$D_a = \frac{99-70\sqrt{2}}{68-48\sqrt{2}} \approx 0.043 \qquad D_{a_1a_2} = \frac{215-152\sqrt{2}}{8(3-2\sqrt{2})^2} \approx 0.17. \tag{28}$$

Thus for equatorial qubits, we can find smaller D_a and $D^{(2)}_{a_1a_2}$, which means that QCM (2) has a higher fidelity than the original UQCM [5] in terms of the Hilbert–Schmidt norm. Actually, because we assume α and β to be real, only a single unknown parameter is copied instead of two unknown parameters for the case of a general pure state. Thus a higher fidelity of quantum cloning can be achieved. The case of spin flip has a similar phenomenon [18, 21].

Under condition (27), we have

$$\xi = \frac{1}{8} \qquad \eta = \frac{\sqrt{2}-1}{12-8\sqrt{2}}. \tag{29}$$

We can realize vectors $|Q_j\rangle, |Y_j\rangle, j = 0, 1$ in two-dimensional space,

$$
\begin{aligned}
|Q_0\rangle &= \left(0, \frac{1}{4-2\sqrt{2}}\right) \qquad & |Q_1\rangle &= \left(\frac{1}{4-2\sqrt{2}}, 0\right) \\
|Y_0\rangle &= \left(\frac{1}{2\sqrt{2}}, 0\right) \qquad & |Y_1\rangle &= \left(0, \frac{1}{2\sqrt{2}}\right).
\end{aligned}
\tag{30}
$$

Transformation (2) is rewritten as

$$
|0\rangle_{a_1}|Q\rangle_{a_2a_3} \to \frac{1}{4-2\sqrt{2}}[|00\rangle_{a_1a_2} + (3-2\sqrt{2})|11\rangle_{a_1a_2}]|\uparrow\rangle_{a_3} + \frac{1}{2}|+\rangle_{a_1a_2}|\downarrow\rangle_{a_3} \tag{31}
$$

$$
|1\rangle_{a_1}|Q\rangle_{a_2a_3} \to \frac{1}{4-2\sqrt{2}}[|11\rangle_{a_1a_2} + (3-2\sqrt{2})|00\rangle_{a_1a_2}]|\downarrow\rangle_{a_3} + \frac{1}{2}|+\rangle_{a_1a_2}|\uparrow\rangle_{a_3}. \tag{32}
$$

This transformation agrees with that obtained by Bruß *et al* [19]. Using BB84 states as input, they showed that this transformation is the optimal cloning transformation for equatorial qubits. This means that the proposed cloning transformation for the *x*–*z* equator (2) indeed realizes the optimal QCM within the Hilbert–Schmidt norm.

For an arbitrary λ with conditions (17) and (24) satisfied, we can still realize vectors $|Q_j\rangle, |Y_j\rangle, j = 0, 1$ in two-dimensional space,

$$
|Q_0\rangle = q|\uparrow\rangle \qquad |Q_1\rangle = q|\downarrow\rangle \qquad |Y_0\rangle = y|\downarrow\rangle \qquad |Y_1\rangle = y|\uparrow\rangle \tag{33}
$$

where we use the notation

$$
q \equiv \sqrt{\frac{2}{3-2\lambda+3\lambda^2}} \qquad y \equiv \frac{1-\lambda}{\sqrt{6-4\lambda+6\lambda^2}}. \tag{34}
$$

Thus all transformations (2) satisfy conditions (I) and (II). Explicitly, the quantum cloning transformation for pure input states (1) can be written as

$$
\begin{aligned}
|0\rangle_{a_1}|Q\rangle_{a_2a_3} &\to (|00\rangle_{a_1a_2} + \lambda|11\rangle_{a_1a_2})q|\uparrow\rangle_{a_3} + (|10\rangle_{a_1a_2} + |01\rangle_{a_1a_2})y|\downarrow\rangle_{a_3} \\
|1\rangle_{a_1}|Q\rangle_{a_2a_3} &\to (|11\rangle_{a_1a_2} + \lambda|00\rangle_{a_1a_2})q|\downarrow\rangle_{a_3} + (|10\rangle_{a_1a_2} + |01\rangle_{a_1a_2})y|\downarrow\rangle_{a_3}.
\end{aligned}
\tag{35}
$$

The distances defined by the Hilbert–Schmidt norm take the form (25).

4. Bures fidelity

For finite-dimensional spaces, the Hilbert–Schmidt norm becomes less appropriate when the dimension increases. The Bures fidelity provides a more exact measurement of the distinguishability of two density matrices. In this section, we will use the Bures fidelity to check the result in the previous section. The fidelity is defined as

$$
F(\rho_1, \rho_2) = \left[\mathrm{Tr}\sqrt{\left(\rho_1^{1/2}\rho_2\rho_1^{1/2}\right)}\right]^2. \tag{36}
$$

The value of F ranges from 0 to 1. A larger F corresponds to a higher fidelity. $F = 1$ means that two density matrices are equal. For a pure state, $\rho_1 = |\Psi\rangle\langle\Psi|$, the fidelity can be defined by an equivalent form $F = \langle\Psi|\rho_2|\Psi\rangle$. We shall use definition (36) in this section.

It is known that a matrix

$$
U = \begin{pmatrix} -\frac{\beta}{\alpha} & \frac{\alpha}{\beta} \\ 1 & 1 \end{pmatrix} \tag{37}
$$

diagonalizes $\rho_a^{(\mathrm{in})}$ [22],

$$\rho_a^{(\mathrm{in})} = U \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U^{-1}. \tag{38}$$

We thus have

$$F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right) = \xi + \frac{(1-2\xi)[2\alpha^4(1-\lambda^2) + 2\alpha^2(\lambda^2-1) + 1]}{1+\lambda^2} + 2\alpha^2(1-\alpha^2)\eta(\lambda+1). \tag{39}$$

We demand that the fidelity be independent of the input state,

$$\frac{\partial}{\partial\alpha^2} F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right) = 0. \tag{40}$$

This gives

$$\eta = \frac{1-\lambda}{1+\lambda^2}(1-2\xi) \tag{41}$$

and we obtain

$$F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right) = \frac{1-\xi+\lambda^2\xi}{1+\lambda^2}. \tag{42}$$

Next, we use the Bures fidelity to evaluate the distinguishability of density operators $\rho_{a_1a_2}^{(\mathrm{out})}$ and $\rho_{a_1a_2}^{(\mathrm{in})} = \rho_{a_1}^{(\mathrm{in})} \otimes \rho_{a_1}^{(\mathrm{in})}$. We have

$$\begin{aligned} F\left(\rho_{a_1a_2}^{(\mathrm{in})}, \rho_{a_1a_2}^{(\mathrm{out})}\right) = {} & \frac{1-2\xi}{1+\lambda^2}[\lambda^2 + \alpha^2(1-\lambda^2)]\alpha^4 + 2\alpha^2(1-\alpha^2)\lambda\frac{1-2\xi}{1+\lambda^2} \\ & + 2\alpha^2(1-\alpha^2)(1-2\xi)\frac{1-\lambda^2}{1+\lambda^2} + 4\alpha^2(1-\alpha^2)\xi \\ & + \frac{1-2\xi}{1+\lambda^2}[\alpha^2(\lambda^2-1)+1](1-\alpha^2)^2. \end{aligned} \tag{43}$$

We again impose the condition

$$\frac{\partial}{\partial\alpha^2} F\left(\rho_{a_1a_2}^{(\mathrm{in})}, \rho_{a_1a_2}^{(\mathrm{out})}\right) = 0 \tag{44}$$

which gives

$$\xi = \frac{(1-\lambda)^2}{2(3-2\lambda+3\lambda^2)}. \tag{45}$$

Thus, we finally have two Bures fidelities for one- and two-mode density operators,

$$F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right) = \frac{5-2\lambda+\lambda^2}{2(3-2\lambda+3\lambda^2)} \tag{46}$$

$$F\left(\rho_{a_1a_2}^{(\mathrm{in})}, \rho_{a_1a_2}^{(\mathrm{out})}\right) = \frac{2}{3-2\lambda+3\lambda^2}. \tag{47}$$

We find that the Hilbert–Schmidt norm and the Bures fidelity lead to the same relations (17), (41) and (24), (45). However, fidelities (46) and (47) do not take the maxima simultaneously which is different from the case of the Hilbert–Schmidt norm. In the region $0 < \lambda < \frac{1}{3}$, for both $F\left(\rho_{a_1a_2}^{(\mathrm{in})}, \rho_{a_1a_2}^{(\mathrm{out})}\right)$ and $F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right)$, we can have a higher fidelity than the original UQCM which corresponds to $\lambda = 0$. This result agrees with the previous result using the Hilbert–Schmidt norm. We use fidelity $F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right)$ to define the quality of the copied equatorial qubits. When $\lambda = 3 - 2\sqrt{2}$, $F\left(\rho_a^{(\mathrm{in})}, \rho_a^{(\mathrm{out})}\right)$ takes its maximum which is the same as in the case of the Hilbert–Schmidt norm. Thus, we have shown that both the Hilbert–Schmidt norm and the Bures fidelity give the same result.

When $\lambda = 3 - 2\sqrt{2}$, $F\left(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}\right)$ takes its maximum,

$$F\left(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}\right)\big|_{\lambda=3-2\sqrt{2}} = \frac{1}{2} + \sqrt{\frac{1}{8}} \approx 0.8536 \tag{48}$$

which is larger than the original UQCM,

$$F\left(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}\right)\big|_{\lambda=0} = \tfrac{5}{6} \approx 0.8333. \tag{49}$$

And we also have

$$F\left(\rho_{a_1a_2}^{(\text{in})}, \rho_{a_1a_2}^{(\text{out})}\right)\big|_{\lambda=3-2\sqrt{2}} = \frac{1}{24 - 16\sqrt{2}} \approx 0.7286 > F\left(\rho_{a_1a_2}^{(\text{in})}, \rho_{a_1a_2}^{(\text{out})}\right)\big|_{\lambda=0} = \frac{2}{3} \approx 0.6667. \tag{50}$$

We remark that the optimal fidelity (48) also agrees with the result obtained by Bruß *et al* [19].

5. Summary and discussions

In studying the optimal UQCM, the condition of orientation invariance of the Bloch vector is generally imposed [8]. Under the symmetry condition (I), the condition of orientation invariance of the Bloch vector is equivalent to condition (II) that the distance between the input density operator and the output density operators is input state independent. We can check that for the case under consideration in this paper, the orientation invariance of the Bloch vector means relation (17) or (41) which is the subsequence of condition (II).

Different from the UQCM, the copied qubits are separable in a phase-covariant cloning machine. We can realize quantum cloning by using the quantum networks. By using the fidelity between the input state and the reduced density operator of output, we can also generalize the cloning to N qubits input and M copies [23], and 1 to 3 cloning and other cases were studied independently in [24].

We have already mentioned that the transformation and optimal fidelity in this paper agree with the results in [19]. However, we use a different method. And we use various criteria to quantify the quality of this cloning machine. This is necessary because, for example, in eavesdropping on quantum cryptography, Eve perhaps needs to compare different distinguishability measures and then to choose a suitable one to use. In this paper, not only are the reduced density operators used to compare the input and output, but also the whole output state is used to compare the ideal output copies in quantifying the quality of the cloning machine. An interesting result is that when we use the Bures fidelity as the quality measure, the one-particle density matrices and two-particle density matrices give different optimal points. When $\lambda = 3-2\sqrt{2}$, $F\left(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}\right)$ takes its maximum, whereas for $\lambda = 1/3$, $F\left(\rho_{a_1a_2}^{(\text{in})}, \rho_{a_1a_2}^{(\text{out})}\right)$ achieves its maximum. This could be useful in eavesdropping on quantum key distribution. If Eve just uses the quantity $F\left(\rho_{a_1a_2}^{(\text{in})}, \rho_{a_1a_2}^{(\text{out})}\right)$ to find the quantum state in quantum cryptography, she can adjust her cloning machine and let $\lambda = 1/3$ so that she can get the optimal result. If both one-particle and two-particle density matrices are used, then $\lambda = 3 - 2\sqrt{2}$ seems better. We should note that for UQCM, if we use the Bures fidelity as the quality measure, the one-particle density matrices and two-particle density matrices give the same optimal point.

Recently, Jozsa proposed the stronger no-cloning theorem which states that if we want to copy a quantum state exactly, the ancilla (blank) state should already have the full information of this quantum state, see [3]. For the case of cloning an equatorial qubit, though the ancilla state does not contain the full information of this equatorial qubit, we can let the ancilla state contain the partial information we already know about this equatorial qubit. This is perhaps the reason why we can have a better copy than UQCM in which we know nothing about the quantum state to be cloned.

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